AP Precalc

Write your questions and thoughts here!

4.3 Parametric Functions and Rates of Change

4.3 Notes

Recall that a parametric function is $f(t) = (x(t), y(t))$, where $x(t)$ is modeling the horizontal component of the particle motion and $y(t)$ is modeling the vertical component of the particle motion.

Direction of Planar Motion

As the parameter t increases, the following conditions apply

- If $\boldsymbol{x(t)}$ **is decreasing**, the particle is moving to the ______.
- If $\boldsymbol{x(t)}$ **is increasing**, the particle is moving to the ______.
- If $\boldsymbol{y(t)}$ **is decreasing**, the particle is moving ______.
- If $\boldsymbol{y(t)}$ **is increasing**, the particle is moving ______.

REMEMBER: This is over an interval of increasing values of the parameter $\boldsymbol{t}$.

Because we are able to model the horizontal and vertical component of the particle motion using $x(t)$ and $y(t)$ respectively, it follows that at any given point in the plane, the direction of planar motion may be different for different values of t. What that means is as the time increases the direction of motion can change. Circles and Parabolas are great examples. Let's look at a simple example to illustrate this.

1. Example of a circle.

A particle's position and motion are modeled by the parametric equation $f(t) = (\cos(t), \sin(t))$ for $0 \le t \le 2\pi$. Below is a table of values for this set of parametric equations. Assume the direction does not change between the values t shown in the table.

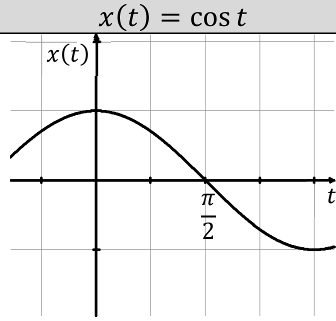
t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
x	1	$\frac{\sqrt{2}}{2}$	0	$-\frac{\sqrt{2}}{2}$	-1	$-\frac{\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1
y	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{2}}{2}$	0	$-\frac{\sqrt{2}}{2}$	-1	$-\frac{\sqrt{2}}{2}$	0

Examine both $x(t)$ and $y(t)$ on the interval $0 \le t \le \frac{\pi}{2}$.

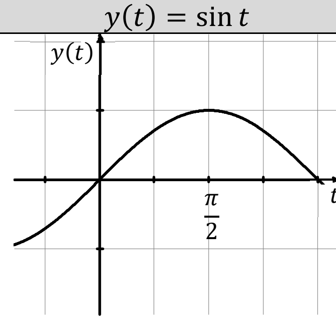
a. Is the particle moving left or right on the interval $0 \le t \le \frac{\pi}{2}$?

b. Is the particle moving up or down on the interval $0 \le t \le \frac{\pi}{2}$?

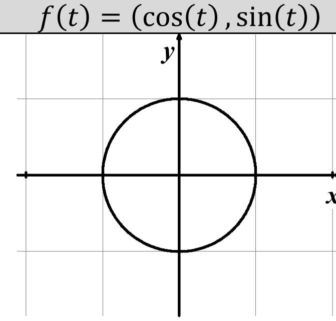
$x(t) = \cos t$



$y(t) = \sin t$



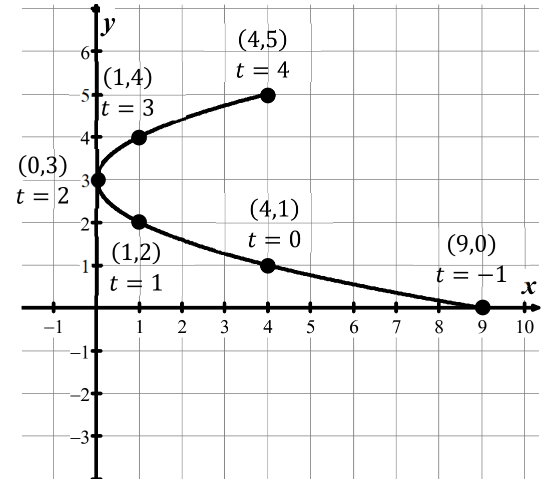
$f(t) = (\cos(t), \sin(t))$



Write your questions and thoughts here!

A Curve can be Parametrized in Different Ways

A particle's position and motion are modeled by the parametric equation $f(x) = ((t-2)^2, t+1)$ for $-1 \leq t \leq 4$. On the right is a graph with the motion labeled.

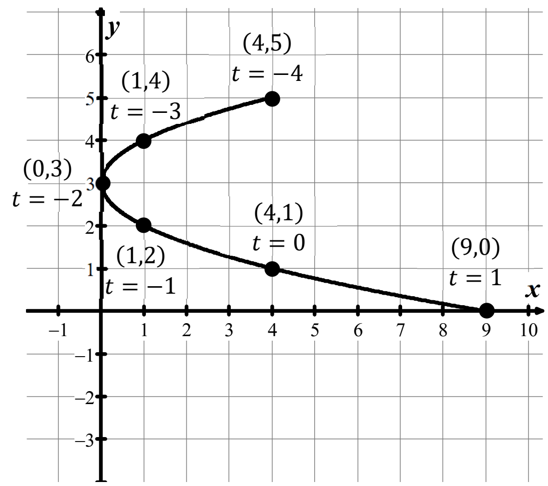


By looking at the x and y values as the values of parameter t are increasing, we are able to determine the following.

- On the interval $-1 \leq t < 2$, the movement is ____________________.
- On the interval $2 \leq t \leq 4$, the movement is ____________________.

A curve can be parametrized in different ways and can be traversed in different directions with different parametric functions. **We will learn how to parametrize a rectangular equation in a later lesson.**

Let's look at the parametric function $f(t) = ((-t-2)^2, 1-t)$ on the restricted domain of $-4 \leq t \leq 1$. The graph is shown on the right.



The graph looks identical to the graph in the previous problem, but there is an important difference. The position and motion have changed.

- On the interval $-4 \leq t < -2$, the movement is ____________________.
- On the interval $-2 \leq t \leq 1$, the movement is ____________________.

These two parametric functions that we've graphed have the same shape but have different directions of motion. Why? **The shape of the graph does not determine the direction of planar motion, it is determined by how the curve has been parametrized.** Think of it this way: imagine a sidewalk. You can walk in either direction on this sidewalk, yet it's the same sidewalk.

Change the Direction of a Parametric Function

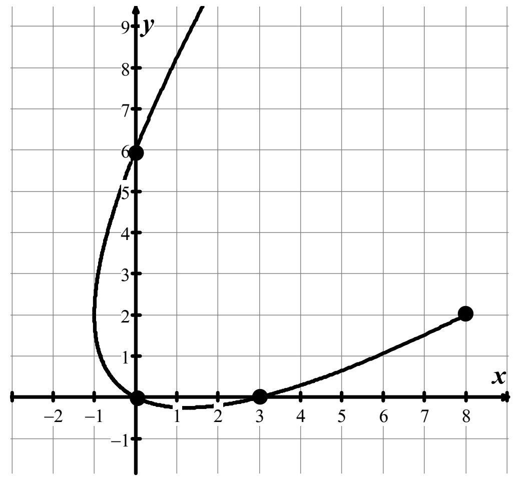
To keep the same curve of a parametric function, but change the direction, we negate the input of the parameter.

$(x(t), y(t))$ has the same curve as __________________ with the opposite direction of motion.

Also, if the domain of the parameter is restricted to $a \leq t \leq b$ then to get the opposite direction of motion we change the restriction to __________________.

Write your questions and thoughts here!

2. Given the parametric function $f(t) = (t^2 - 2t, t^2 + t)$ on the interval $-2 \le t \le 3$. Create a parametric function $g(t)$ that gives the same curve, but the motion is in the opposite direction.



Average Rate of Change

Recall from algebra, the slope (rate of change) of any line is… $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{rise}{run}$

We can do a similar thing for parametric equations.

Find the slope between two points on a parametric curve

Over a given interval $[t_1, t_2]$ within the domain, the average rate of change can be computed for $x(t)$ and $y(t)$ independently.

$$\frac{\Delta x}{\Delta t} = \_\_\_\_\_\_\_\_\_\_ \quad \text{and} \quad \frac{\Delta y}{\Delta t} = \_\_\_\_\_\_\_\_\_\_$$

The ratio of the avg. rate of change of y to avg. rate of change of x gives the slope of the line between the points on the curve corresponding to t_1 and t_2.

Slope = ——

The average rate of change of $x(t) \ne 0$.

3. Use the parametric function $f(t) = (2t, \sqrt{9 - t^2})$ for $-3 \le t \le 0$ to find the following.
   a. The average rate of change of $y(t)$ on $-3 \le t \le 0$.

   b. The average rate of change of $x(t)$ on $-3 \le t \le 0$.

   c. The slope of the line between the points found using $t = -3$ and $t = 0$.

The table below is a numerical table of values that describe the location of a particle over the interval $0 \leq t \leq 8$.

t	0	1	2	3	4	5	6	7	8
x	2	3	4	5	6	7	8	9	10
y	30	16	6	0	-2	0	6	16	30

Use the table to answer questions 1-5.

1. Find the average rate of change of $x(t)$ over the interval $1 \leq t \leq 4$.

2. Find the average rate of change of $y(t)$ over the interval $6 \leq t \leq 8$.

3. Over which interval is the direction of particle motion to the right and upward? Assume the direction does not change between the values t shown in the table.

4. Over which interval is the direction of particle motion to the right and downward? Assume the direction does not change between the values t shown in the table.

5. Find the slope of the graph between the points on the graph that correspond to the points that correspond to $t = 4$ and $t = 7$.

Find a set of parametric equations that will produce the same path as the given set of equations with a direction of particle motion in the opposite direction. Use graphing technology to verify your answer.

6. $x(t) = -2(t+2)^2 + 4$, $y(t) = 3t - 4$ for $-3 \leq t \leq 1$.

7. $x(t) = \sqrt{4-t}$, $y(t) = t + 1$ for $2 \leq t \leq 4$.

8. $x(t) = |t+5| + 1,\ y(t) = 3 - t$ for $-5 \le t \le -1$.

9. $x(t) = \frac{t-1}{2},\ y(t) = \frac{2}{3}(t-2)(t+4)(1-t)$ for $-8 \le t \le 2$.

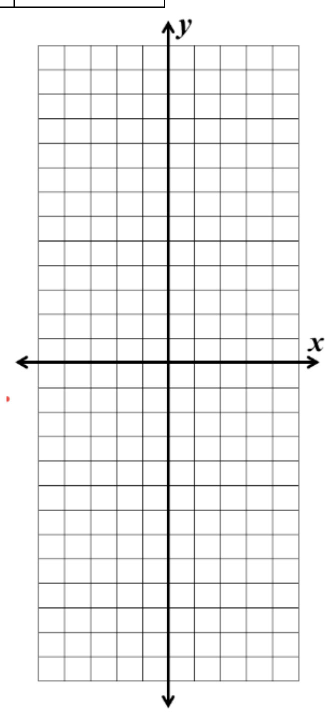
10. $x(t) = t + 1,\ y(t) = t^2 - 4t + 1$ for $-10 \le t \le -3$.

11. A particles motion in the plane is modeled by the parametric $x(t) = -\frac{1}{2}(t-2)^2$ and $y(t) = 2t + 3$.

a. Fill in the table of values and use it to describe the direction of the particle motion over the interval $-1 \le t \le 5$. Assume the direction does not change between the values t shown in the table.

t	-1	0	1	2	3	4	5
x							
y							

b. Sketch a graph and indicate the direction of movement along the path.



Find the slope of the line between the points that correspond to the given values of t.

12. $f(t) = (t + 1, 2t + 1)$ at $t = 1$ and $t = 5$

13. $f(t) = (-2|t + 4| - 3, t)$ at $t = -2$ and $t = 4$